Journal of Approximation Theory **110**, 120–124 (2001)
doi:10.1006/jath.2000.3554, available online at http://www.idealibrary.com on IDEAL®

A Lower Estimate for Entropy Numbers

Thomas Kühn

Fakultät für Mathematik und Informatik, Mathematisches Institut, Universität Leipzig, Augustusplatz 10/11, 04109 Leipzig, Germany
E-mail: kuehn@mathematik.uni-leipzig.de

Communicated by Allan Pinkus

Received June 28, 2000; accepted in revised form November 6, 2000; published online February 16, 2001

The behaviour of the entropy numbers $e_k(id\colon l_p^n \to l_q^n)$, $0 < p < q \leqslant \infty$, is well known (up to multiplicative constants independent of n and k), except in the quasi-Banach case $0 < p < 1$ for "medium size" k, i.e., when $\log n \leqslant k \leqslant n$, where only an upper estimate is available so far. We close this gap by proving the lower estimate $e_k(id\colon l_p^n \to l_q^n) \geqslant c(\log(n/k+1)/k)^{1/p-1/q}$ for all $0 < p < q \leqslant \infty$ and $\log n \leqslant k \leqslant n$, with some constant $c > 0$ depending only on p. © 2001 Academic Press

Key Words: entropy numbers; sequence spaces.

Let $T\colon X \to Y$ be a (bounded linear) operator from a quasi-Banach space X into another quasi-Banach space Y. The kth (dyadic) entropy number of T is defined as

$$e_k(T) = \inf\{\varepsilon > 0 : T(B_X) \text{ can be covered by } 2^{k-1} \text{ balls of radius } \varepsilon \text{ in } Y\},$$

where B_X denotes the closed unit ball in X. Entropy numbers are closely related to the concept of metric entropy developed by Kolmogorov in the 1930s. For the basic properties of entropy numbers and their use in applications to eigenvalue and compactness problems we refer to the monographs by König [5], Pietsch [6], Carl and Stephani [1], Edmunds and Triebel [2], Triebel [8], and the references given therein.

In many applications certain discretization techniques are used, which allow to reduce an infinite dimensional problem to a finite dimensional one. Therefore it is essential to have precise information on the entropy numbers of, say, the identity operators $id\colon l_p^n \to l_q^n$, $0 < p \leqslant q \leqslant \infty$. Here l_p^n stands as usual for the space $\mathbb{R}^n$ equipped with the quasi-norm $\|x\|_p = (\sum_{k=1}^n |x_k|^p)^{1/p}$ for $0 < p < \infty$ and $\|x\|_p = \sup_{1 \leqslant k \leqslant n} |x_k|$ for $p = \infty$, which is even a norm for $p \geqslant 1$. In the sequel, all constants are positive real numbers which may depend on the parameters p and q but not on the

0021-9045/01 $35.00

integers n and k. Logarithms are taken with respect to the base 2, $\log = \log_2$.

Schütt [7] proved in 1984 (see also [5, 3.c.8]) for $1 \leqslant p \leqslant q \leqslant \infty$ the inequality

$$e_k(id\colon l_p^n \to l_q^n) \leqslant c \cdot \begin{cases} 1 & \text{if } \; 1 \leqslant k \leqslant \log n \\ \left(\dfrac{\log(n/k+1)}{k}\right)^{1/p-1/q} & \text{if } \; \log n \leqslant k \leqslant n \\ 2^{-(k-1)/n} n^{1/q-1/p} & \text{if } \; k \geqslant n. \end{cases} \tag{1}$$

He also showed that this is sharp, meaning that there is a similar lower estimate for the entropy numbers (of course, with another constant). Next, in 1996 Triebel and Edmunds [2, Proposition 3.2.2] extended Schütt's upper estimate to the quasi-Banach case $0 < p \leqslant q \leqslant \infty$, and finally in 1997 Triebel [8, Theorem 7.3] showed that the lower estimate also remains valid in this case, provided k is either "small" ($1 \leqslant k \leqslant \log n$) or "large" ($k \geqslant n$). For "medium" k no nontrivial lower estimate seems to be known. We will now close this gap.

In the proof of the following theorem we use the Hamming distance on a suitable set. This idea comes from the recent paper by Gao [3], I am grateful to J. Creutzig for drawing my attention to Gao's article. Moreover, I am indebted to an anonymous referee of this note for pointing out a more standard proof via interpolation, a sketch will be given in Remark 3. However, our direct combinatorial proof has the advantage that it works for all possible values of the parameters $0 < p < q \leqslant \infty$ simultaneously.

THEOREM. *Let $0 < p < q \leqslant \infty$. Then there is a constant $c > 0$ such that for all $n, k \in \mathbb{N}$ with $\log n \leqslant k \leqslant n$ the lower estimate*

$$e_k(id\colon l_p^n \to l_q^n) \geqslant c \left(\frac{\log(n/k+1)}{k}\right)^{1/p-1/q} \tag{2}$$

holds.

Proof. For arbitrary integers $n, m \in \mathbb{N}$ with $n \geqslant 4$ and $1 \leqslant m \leqslant \frac{n}{4}$ consider the set

$$S := \left\{ x = (x_j) \in \{-1, 0, 1\}^n : \sum_{j=1}^{n} |x_j| = 2m \right\}.$$

Obviously S has cardinality

$$\# S = \binom{n}{2m} \cdot 2^{2m}$$

and the set $(2m)^{-1/p} S$ is contained in the unit sphere of l_p^n. Denote by h the Hamming distance on S,

$$h(x, y) = \#\{j : x_j \neq y_j\}.$$

Then it is easily verified that, for every fixed $x \in S$,

$$\#\{y \in S : h(x, y) \leqslant m\} \leqslant \binom{n}{m} \cdot 3^m.$$

Indeed, all elements $y \in S$ with $h(x, y) \leqslant m$ can be obtained as follows. First take any subset $J \subseteq \{1, ..., n\}$ of cardinality m, then set $y_j := x_j$ for $j \notin J$ and choose $y_j \in \{-1, 0, 1\}$ arbitrarily for $j \in J$.

Now let $A \subseteq S$ be any subset of S of cardinality at most $a := \binom{n}{2m} / \binom{n}{m}$. The estimate

$$\begin{aligned} \#\{y \in S : \exists x \in A \text{ with } h(x, y) \leqslant m\} &\leqslant \# A \cdot \binom{n}{m} \cdot 3^m \\ &\leqslant \binom{n}{2m} \cdot 3^m < \# S \end{aligned}$$

shows that one can find an element $y \in S$ with $h(x, y) > m$ for all $x \in A$. Therefore one can inductively construct a subset $A \subseteq S$ with $\# A > a$ and the property $h(x, y) > m$ for any two distinct elements $x, y \in A$, whence $\|x - y\|_q > m^{1/q}$. Therefore we have found a subset of the unit ball of l_p^n, namely the set $(2m)^{-1/p} A$, of cardinality larger than a, whose elements have mutual l_q^n-distance $\|x - y\|_q > \varepsilon := (2m)^{-1/p} m^{1/q}$. This implies, if we set $k := [\log a]$, that

$$e_k(id\colon l_p^n \to l_q^n) \geqslant \varepsilon/2 = c_1 m^{1/q - 1/p},$$

where the constant $c_1 = 2^{-(1 + 1/p)}$ depends only on p. By definition of a we have

$$a = \frac{\binom{n}{2m}}{\binom{n}{m}} = \frac{m!\,(n-m)!}{(2m)!\,(n-2m)!} = \prod_{j=1}^{m} \frac{n - 2m + j}{m + j},$$

and since the function $f(x)=\frac{n-2m+x}{m+x}$ is decreasing for $x>0$, it follows that $(\frac{n-m}{2m})^m \leqslant a \leqslant (\frac{n-2m}{m})^m$. Therefore we get

$$c_2 m \log \frac{n}{m} \leqslant k \leqslant m \log \frac{n}{m}$$

with some constant $c_2>0$ independent on n and m. Finally observe that the function $y=x \log \frac{n}{x}$ is strictly increasing on $[1, \frac{n}{4}]$ and maps this interval onto $[\log n, \frac{n}{2}]$, whence its inverse function exists on the latter interval, and it is easy to check that then $x \geqslant \frac{y}{\log(n/y)}$. This shows that

$$e_k(id\colon l_p^n \to l_q^n) \geqslant c \left(\frac{\log(n/k+1)}{k} \right)^{1/p-1/q} \qquad \text{whenever} \qquad \log n \leqslant k \leqslant \frac{c_2 n}{2}.$$

For $c_2 n/2 \leqslant k \leqslant n$ this inequality follows from the monotonicity of entropy numbers and the lower estimate $e_n(id\colon l_p^n \to l_q^n) \geqslant c n^{1/q-1/p}$. The proof is finished.

Remark 1. It is obvious from the proof that (2) holds for complex spaces, too.

Remark 2. The constant c in (2) may in fact be chosen independently of q, as the proof shows. The same is true for the upper estimate (1); see [2, Remark 2, p. 101].

Remark 3. Finally we sketch the above mentioned alternative (more standard) proof of the theorem via interpolation, following an observation of a referee. The well-known behaviour under interpolation of entropy numbers of operators between *Banach* spaces (see, e.g., [6, Chap. 12]) has recently been extended in [4, Sect. 3.2] to operators between *quasi-Banach* spaces. This allows to derive the case $0<p<1$ in the lower estimate (2) from the known cases in (1) and (2). For simplicity of notation let $id_{p,q}^n$ be the identity from l_p^n into l_q^n, moreover we set $f(n,k) := \frac{\log(n/k+1)}{k}$.

In a first step let $1<q \leqslant \infty$, and define $0<\theta<1$ by $1=\frac{1-\theta}{q}+\frac{\theta}{p}$. Using the lower estimate (2) for the identity $id_{1,q}^n$ we obtain by interpolation, for arbitrary integers $n, k \in \mathbb{N}$ with $\log n \leqslant k \leqslant n$,

$$c_1 f(n,k)^{1-1/q} \leqslant e_k(id_{1,q}^n) \leqslant c_2 \| id_{q,q}^n \|^{1-\theta} e_k(id_{p,q}^n)^\theta = c_2 e_k(id_{p,q}^n)^\theta,$$

where the constants c_i are independent of k and n. Observing that $1-\frac{1}{q}=\theta(\frac{1}{p}-\frac{1}{q})$ this implies the desired lower estimate

$$e_k(id_{p,q}^n) \geqslant c f(n,k)^{1/p-1/q}.$$

In a second step we consider the case $0 < q \leqslant 1$, where we define $0 < \eta < 1$ by $\frac{1}{2} = \frac{1-\eta}{\infty} + \frac{\eta}{q}$. The just proved lower estimate for the identity $id^n_{p,2}$, the known upper estimate for $id^n_{p,\infty}$, and interpolation imply

$$c_3\, f(n,k)^{1/p-1/2} \leqslant e_{2k}(id^n_{p,2}) \leqslant c_4\, e_k(id^n_{p,\infty})^{1-\eta}\, e_k(id^n_{p,q})^{\eta}$$
$$\leqslant c_5\, f(n,k)^{(1-\eta)/p}\, e_k(id^n_{p,q})^{\eta},$$

again with constants c_i independent of k and n. Since $\frac{1}{p} - \frac{1}{2} - \frac{1-\eta}{p} = \eta\,(\frac{1}{p} - \frac{1}{q})$, this yields the lower estimate $e_k(id^n_{p,q}) \geqslant c\, f(n,k)^{1/p-1/q}$ in this case, too.

REFERENCES

1. B. Carl and I. Stephani, "Entropy, Compactness and Approximation of Operators," Cambridge Univ. Press, Cambridge, UK, 1990.
2. D. E. Edmunds and H. Triebel, "Function Spaces, Entropy Numbers and Differential Operators," Cambridge Univ. Press, Cambridge, UK, 1996.
3. F. Gao, Metric entropy of convex hulls, preprint.
4. D. D. Haroske and H. Triebel, Entropy numbers in weighted function spaces and eigenvalue distribution of some degenerate pseudodifferential operators, I, *Math. Nachr.* **167** (1994), 131–156.
5. H. König, "Eigenvalue Distribution of Compact Operators," Birkhäuser, Basel, 1986.
6. A. Pietsch, "Operator Ideals," Deutscher Verl. Wissenschaften, Berlin, 1978; North-Holland, Amsterdam, 1980.
7. C. Schütt, Entropy numbers of diagonal operators between symmetric Banach spaces, *J. Approx. Theory* **40** (1984), 121–128.
8. H. Triebel, "Fractals and Spectra," Birkhäuser, Basel, 1997.

Printed in Belgium